ARTICLE NO. 80-4005

G80-012

# Minimal and Subminimal Simplification

0016
0003

Rudrapatna V. Ramnath*
*The Charles Stark Draper Laboratory, Inc., Cambridge, Mass.*

**Analysis of complex dynamic systems usually necessitates the use of approximations, which are often based on intuition. In this paper the rationale underlying asymptotic approximations is discussed in the context of problem solving. In particular, the principle of minimal simplification and its extension, subminimal simplification, are discussed. These principles are useful in obtaining a proper ordering of the different terms appearing in an equation containing a small parameter. As examples, we consider the problem of approximately factoring a quintic polynomial and solving a nonlinear differential equation.**

## Introduction

IN the approximate analysis of dynamic systems described by ordinary differential equations, it is often not clear as to how to simplify the system in deriving the approximations. A very useful approach is through asymptotic analysis which was initiated by Poincare,[1] Stieltjes,[2] and others. However, one seldom finds that a problem is directly amenable to asymptotic techniques so that resort to ad hoc methods and intuition must be made to derive approximations. As a consequence, Kruskal[3] introduced "asymptotology" as a subject that is more than asymptotics: one that includes both the art and science of asymptotic analysis. In his paper, Kruskal presented several asymptotological principles into which some basic ideas of approximation and problem solving have been condensed.

There are many problems in flight dynamics analysis and control which can be fruitfully studied by means of the asymptotological techniques. However, they seem to be relatively unknown in these disciplines—perhaps because Kruskal's work appears in the literature on plasma physics. As one who has used these principles and extended them in the context of several problems in flight dynamics and control, the present author feels that they would be of benefit to flight dynamicists and control engineers in particular and the aerospace community in general. With this motivation, some asymptotological principles with examples are presented—in particular, Kruskal's principle of minimal simplification with an extension by Ramnath,[4-6] the principle of subminimal simplification, to study problems for which the original method is not amenable.

## Principle of Minimal Simplification

Simplification of a mathematical system requires neglecting terms with a resultant loss of information. Kruskal's principle states that the system must be simplified the least and yet allow us to develop approximations which have the widest applicability. Often, however, two or more such maximal sets of terms exist such that no set includes all terms of any other. Each set corresponds to a different asymptotic behavior. The correctness of a particular ordering is usually checked for self-consistency. That is, the approximate solution does not violate the assumption under which it was derived.

For example, consider the problem of finding the roots of the quintic equation

$$0.0001\,s^5 + 0.0002\,s^4 + s^3 + s^2 + 0.03\,s + 0.04 = 0 \qquad (1)$$

which might be the characteristic equation arising in the analysis of a flight control system. Since a general solution is not practical, we attempt to develop approximate solutions using the fact that some of the coefficients are "small." The equation is first parameterized by a parameter $\epsilon = 0.01$ as

$$\epsilon^2 s^5 + 2\epsilon^2 s^4 + s^3 + s^2 + 3\epsilon s + 4\epsilon = 0 \qquad (2)$$

We could begin, of course, by neglecting all terms containing $\epsilon$ [this implies an assumption that $s = O(1)$] to obtain a nontrivial value for $s$ as $s = -1$ and which is consistent with the assumption. If there are many terms, such a procedure—involving an assumption on the magnitude of the solution, neglecting nonessential terms, and checking the resulting approximation for consistency—becomes tedious indeed.

A systematic approach would lead to the dominant terms determining the correct approximation. We see that no matter which terms are dominant, $s$ will behave predominantly as some power of $\epsilon$. Therefore, we assume a general form

$$s = \epsilon^m y \qquad y = O(1) \qquad (3)$$

Equation (2) now becomes

$$\epsilon^{(2+5m)} y^5 + 2\epsilon^{(2+4m)} y^4 + \epsilon^{3m} y^3 + \epsilon^{2m} y^2 + 3\epsilon^{(1+m)} y + 4\epsilon = O \qquad (4)$$

The desire to balance at least two terms helps determine the choice of $m$. Pairwise balancing of terms in Eq. (4) yields $\binom{6}{2}$ relations leading to nine different values for $m$ given by

$$m = -2,\ -1,\ -\tfrac{2}{3},\ -\tfrac{1}{4},\ -1/5,\ 0,\ \tfrac{1}{3},\ \tfrac{1}{2},\ 1 \qquad (5)$$

Substituting these values for $m$ successively into Eq. (4) and neglecting small terms, we see that for consistency, $m$ must be chosen to be

$$m = 0,\ -1,\ \tfrac{1}{2} \qquad (6)$$

Thus, only the choice of $m$ given by Eq. (6) balances the maximum number of dominant terms and the system simplifies the least.

This technique has a very useful graphical interpretation. Each term of Eq. (4) is of the general form $\epsilon^{a+bm}$ with $a$ and $b$ taking on different integer values. Each term is represented as

Received April 24, 1978; revision received June 22, 1979. 

Index categories: Analytical and Numerical Methods; Guidance and Control.

*Leader, Division of Education; also Lecturer, Dept. of Aeronautics & Astronautics, Dept. of Mechanical Engineering, MIT. Associate Fellow AIAA.

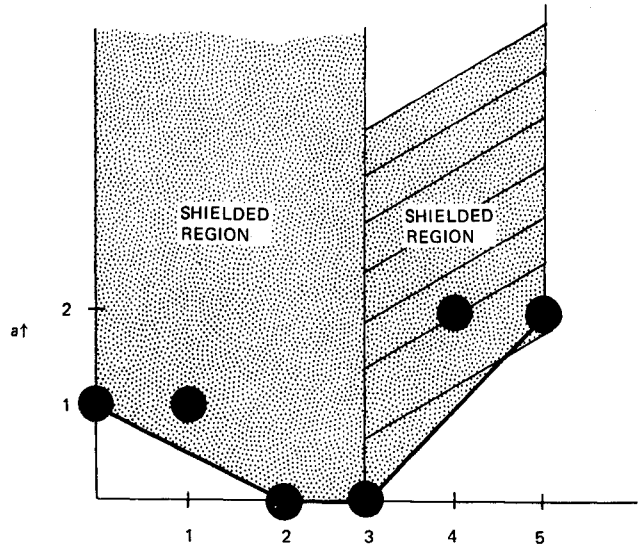


**Fig. 1 Minimal simplification.**

a point on the graph of Fig. 1 where $a$ is plotted along the abscissa (power of $s$) and $b$ is along the ordinate (power of $\epsilon$). The coefficients, which are assumed to be O(1), are ignored and the equation is represented by a cluster of points. For a specific value of $m$, the relationship between $s$ and $\epsilon$ corresponds to the asymptotic behavior of all terms represented by points lying on a common line with a definite slope. This is true whether some terms are present or not. For example, all terms lying on a line with a slope of $-1$ correspond to a common asymptotic behavior of $s \sim \epsilon$. Similarly, all points on a line of slope $+1$ correspond to $s \sim 1/\epsilon$. As $a$ increases positively, the magnitude of the term decreases.

In general, we seek a relation balancing two terms for the smallest possible value of $a$. This corresponds to the lowest convex support of the cluster of graphed points which can be obtained by pushing a line (or a ruler) from below until it passes through a graphed point, then rotating it about that point until it passes through another graphed point, such that there are no graphed points below the line. This constitutes the lower convex support[3] and corresponds to the proper (i.e., best balanced) asymptotic ordering for small $\epsilon$. This describes the dominant or leading order behavior. We note, however, that the rotation of the ruler in either sense must be such that the support line is *not* vertical. A transformation is then made to render the terms corresponding to points on a support line to be of the same order. A perturbation expansion then will enable us to generate more accurate approximations. When there is more than one balance (or support) line, each corresponds to a different asymptotic behavior corresponding to different solutions.

In Fig. 1 there are three such lines. The upslanting 45-deg line on the right balances the terms $\epsilon^2 s^5$ and $s^3$ leading to the root $s = \pm i/\epsilon$ where $i = \sqrt{-1}$. The horizontal support line joining points (2,0) and (3,0) balances the terms $s^2$ and $s^3$ and yields $s = -1$. The upslanting line on the left connecting points (1,0) and (2,0) balances the $s^2$ and $4\epsilon$ terms giving $s = \pm 2i\sqrt{\epsilon}$. Thus, the leading approximations for the roots are

$$s = \pm i/\epsilon, \quad -1, \quad \pm 2i\sqrt{\epsilon} \tag{7}$$

To derive this result using all of the different values for $m$ as outlined earlier would indeed be quite laborious. The minimal simplification approach is therefore very useful.†

†The exact roots of Eq. (1) are $-0.4999520 \pm 99.99359i$, $-1.009627$, and $0.004765891 \pm 0.1989973i$.

Another important observation is that some terms may be rejected once and for all from the competition. In the present example, the term $2\epsilon^2 s^4$ is smaller (i.e., of higher order) than $\epsilon^2 s^5$ or $s^3$ no matter how $s$ varies with $\epsilon$, i.e, whatever the value of $m$. Therefore, it will never contribute to the leading behavior of a root. Maximal balancing thus enables us to identify a "shielded" region (shaded region in Fig. 1) formed by the points (5,2) and (3,0). The point (4,2), corresponding to the term $2\epsilon^2 s^4$, cannot be on a support line. Similarly any number of points lying in this semi-infinite shielded strip may be ignored in determining the dominant behavior of the roots. Another such region shielded by the points (0,1) and (2,0) can also be identified (dotted area in Fig. 1). Here again, any number of terms which lie in the shielded region may be excluded from competition as they will not affect the dominant behavior of the system. In practice this feature is an extremely useful aid. This approach has been extended to study large-scale systems in which the coefficients are known only numerically.[7]

## Principle of Subminimal Simplification

Quite often, in applied problems, it happens that strict adherence to the principle of minimal simplification does not allow meaningful approximations to be derived. In such cases, subminimal simplification, i.e., minimal simplification of the next rank, helps determine the proper ordering leading to a nontrivial approximation. This is an extension of Kruskal's Minimal Simplification Principle, due to Ramnath.[4,5] What is required in such cases is a compromise between completeness and simplicity. Thus, as in the previous example, one establishes balance lines as before, but instead of no terms appearing below the support line, *one* term is allowed to lie under this line. Then terms corresponding to points on the line constitute a balancing of the next order and enable us to develop approximations which improve upon direct perturbation and yet can be developed rather simply. This principle is particularly useful in nonlinear or time-varying differential equations when solved by the method of multiple scales[4-6] or other advanced methods.

The new principle[4,6] is stated as:

> If application of the Minimal Simplification Principle leads to either the most complete system or to direct perturbation, then Subminimal Simplification leads to useful asymptotic approximations.

This is really a compromise between completeness and simplicity. The most complete and informative system may not be solvable. At the other extreme, a direct perturbation expansion may not be very useful, as it is often nonuniform.

For example, consider a nonlinear damped oscillator described by

$$\frac{d^2y}{dt^2} + \epsilon \frac{dy}{dt}\left|\frac{dy}{dt}\right| + \alpha^2 y = 0 \tag{8}$$

where $\epsilon$ and $\alpha$ are constants and $0 < \epsilon \ll 1$ and $\alpha > 0$ but not small. Using the method of multiple scales, we extend time into two new scales:

$$t \to \{\tau_0, \tau_1\} \qquad \tau_0 = t \qquad \tau_1 = \epsilon^m \int^t k(s)\,ds \tag{9}$$

$$y(t,\epsilon) \to Y(\tau_0, \tau_1, \epsilon)$$

where $k(t)$ is an as yet undetermined clock function and $m$ is a constant which describes the order of smallness separating the time scales. Our goal is to provide a proper rationale by which both $m$ and $k(t)$ can be determined systematically, in developing a correct asymptotic approximation of the solution.

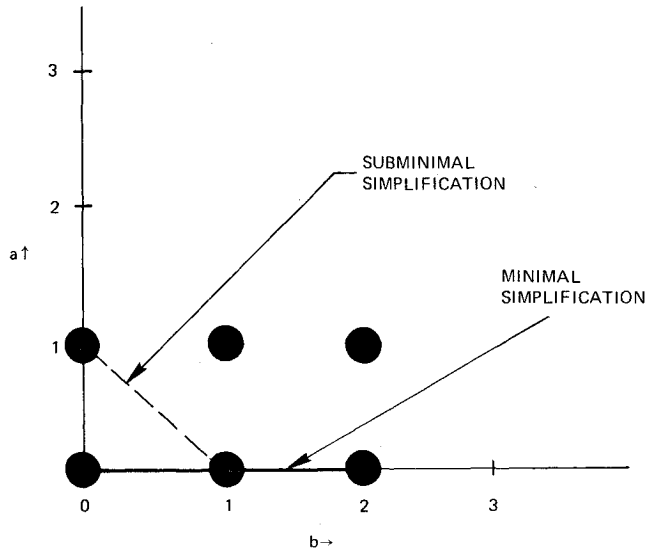


**Fig. 2 Subminimal simplification for nonlinear oscillator.**

The extended form of Eq. (8) becomes

$$\left[\frac{\partial^2 Y}{\partial\tau_0^2}+\alpha^2 Y\right]+\epsilon^m\left[\dot{k}\frac{\partial Y}{\partial\tau_1}+2k\frac{\partial^2 Y}{\partial\tau_0\partial\tau_1}\right]+\epsilon^{2m}\left[k^2\frac{\partial^2 Y}{\partial\tau_1^2}\right]$$
$$+\epsilon\left[\frac{\partial Y}{\partial\tau_0}+\epsilon^m k\frac{\partial Y}{\partial\tau_1}\right]\left|\frac{\partial Y}{\partial\tau_0}+\epsilon^m k\frac{\partial Y}{\partial\tau_1}\right|=0 \tag{10}$$

Now, assuming that $m>0$, we can show that

$$\left|\frac{\partial Y}{\partial\tau_0}+\epsilon^m k\frac{\partial Y}{\partial\tau_1}\right|=\left|\frac{\partial Y}{\partial\tau_0}\right|+\epsilon^m k\frac{\partial Y}{\partial\tau_1}\operatorname{sgn}\left(\frac{\partial Y}{\partial\tau_0}\right) \tag{11}$$

Using this relation and

$$\frac{\partial Y}{\partial\tau_0}\operatorname{sgn}\left(\frac{\partial Y}{\partial\tau_0}\right)=\left|\frac{\partial Y}{\partial\tau_0}\right|$$

in Eq. (10), we have

$$\left[\frac{\partial^2 Y}{\partial\tau_0^2}+\alpha^2 Y\right]+\epsilon^m\left[\dot{k}\frac{\partial Y}{\partial\tau_1}+2k\frac{\partial^2 Y}{\partial\tau_0\partial\tau_1}\right]+\epsilon^{2m}\left[k^2\frac{\partial^2 Y}{\partial\tau_1^2}\right]$$
$$+\epsilon\left[\frac{\partial Y}{\partial\tau_0}\left|\frac{\partial Y}{\partial\tau_0}\right|\right]+\epsilon^{1+m}\left[2\left|\frac{\partial Y}{\partial\tau_0}\right|k\frac{\partial Y}{\partial\tau_1}\right]$$
$$+\epsilon^{1+2m}\left[k^2\left(\frac{\partial Y}{\partial\tau_1}\right)^2\operatorname{sgn}\left(\frac{\partial Y}{\partial\tau_0}\right)\right]=0 \tag{12}$$

A proper ordering of the various terms in this equation is determined by the value of $m$. To find an appropriate choice for $m$, we invoke concepts of minimal simplification. Again, to facilitate the procedure for determining $m$, we represent each term as a graphed point. Each term has the general form $\epsilon^{a+bm}$, $a$ and $b$ taking on different values. As before, plotting $a$ along the ordinate and $b$ along the abscissa, we have the graph illustrated in Fig. 2. Now the lower convex support line, which corresponds to Kruskal's minimal simplification, is the line $a=0$. This means that $m=0$ and the terms generated by the extension in Eq. (10) are all of the same order. Therefore, if we insist on minimal simplification the result is one of straight perturbation with no ordered scaling at all.

Alternatively, we resort to subminimal simplification and seek a subminimal support line with one graphed point lying beneath it. This balances the terms in the second and fourth parentheses in Eq. (12) which dictates a choice of $m=1$. With this value the *counterterms*[8] introduced by multiple scales, i.e., the terms with the clock function $k(t)$, are now determined by the nonlinear damping of the oscillator. Hence, the extended perturbation equations, to first order, are

$$\frac{\partial^2 Y}{\partial\tau_0^2}+\alpha^2 Y=0 \tag{13}$$

$$\dot{k}\frac{\partial Y}{\partial\tau_1}+2k\frac{\partial^2 Y}{\partial\tau_0\partial\tau_1}+\frac{\partial Y}{\partial\tau_0}\left|\frac{\partial Y}{\partial\tau_0}\right|=0 \tag{14}$$

Solving Eq. (13), we have

$$Y(\tau_0,\tau_1)=A_1(\tau_1)\exp(i\alpha\tau_0)+A_2(\tau_1)\exp(-i\alpha\tau_0) \tag{15}$$

and substituting in Eq. (14), we obtain

$$(\dot{k}_1+2i\alpha k_1)\exp(i\alpha\tau_0)\frac{dA_1}{d\tau_1}$$
$$+i\alpha\exp(i\alpha\tau_0)A_1\left|i\alpha\exp(i\alpha\tau_0)A_1\right|=0$$

We can write this last equation in the form

$$\frac{1}{-A_1|A_1|}\frac{dA_1}{d\tau_1}=\frac{\alpha^2|e^{i\alpha\tau_0}|}{\dot{k}_1+2i\alpha k_1}=1 \tag{16}$$

so that the term on the left-hand side depends on $\tau_1$ only while the right-hand side is a function of $\tau_0$ only. Each is set equal to unity without loss of generality. Such a separation of variables enables us to solve for $A_1(\tau_1)$ and the clock $k_1(\tau_0)$ as

$$A_1(\tau_1)=1/(c_1+\tau_1)\qquad k_1(t)=(\alpha/2)+b_1\exp(-2i\alpha t) \tag{17}$$

where $b_1$, $c_1$ are arbitrary constants.

We now restrict the extended solution of Eq. (15) using Eqs. (9) and (17). The asymptotic solution to first order can therefore be written as

$$\tilde{y}_1(t,\epsilon)=Y_1(\tau_0,\tau_1)\left|_{\substack{\tau_0(t)\\ \tau_1(t,\tau)}}\right.=\frac{\exp(i\alpha t)}{c+(\epsilon t/2)+b\exp(-i\alpha t)} \tag{18}$$

where $c$ and $b$ are arbitrary constants related to $c_1$ and $b_1$. Similarly, using the second solution of Eq. (15) in Eq. (13) we obtain $k_2$ as the complex conjugate of $k_1$ and the constants $b_2$, $c_2$. $\tilde{y}_2(t,\epsilon)$ is obtained as the complex conjugate of $\tilde{y}_1(t,\epsilon)$. The arbitrary constant $b$ appears solely through integration of the clock equation. It may be set equal to zero unless we wish to use it to satisfy some other conditions. (Appearance of such constants is another degree of freedom in the multiple scales method and is usually observed while solving the nonlinear equations by this method.) With $b=0$, the solutions are

$$\tilde{y}=\frac{\exp(\pm i\alpha t)}{C+\epsilon t/2} \tag{19}$$

which shows the *correct decay* of the amplitude of the oscillation. Thus, subminimal simplification leads to a correct balance of terms and yields useful approximations.

## Summary and Conclusions

Summarizing, the problem of developing approximations to physical problems has been discussed. Two approaches of deriving simpler representations of systems are presented as embodied in the principles of minimal and subminimal simplification. Illustrative examples are presented.

In dealing with a complex problem, these principles provide a systematic way of deciding which terms in an equation are indeed negligible and thereby facilitate the derivation of simple approximations. In the problems dealing with large-scale systems such as those with many rapid and slow modes, this approach systematically leads to an *ordered* hierarchy of subsystems. In applications, the principle of minimal simplification is actually applied first. In cases where this does not lead to useful solutions, yielding either a system which is complete and cannot be solved or one which corresponds to straight perturbation (which is usually nonuniform and therefore not very useful), then one may apply the principle of subminimal simplification. When used in conjunction with the technique of generalized multiple scales,[5] this approach leads to additional degrees of freedom through new arbitrary constants in the approximation, which must be treated judiciously.

## References

[1] Poincare, H., *Les Methodes Nouvelles de la Mecanique Celeste,* Dover, New York.

[2] Erdelyi, A., *Asymptotic Expansions,* Dover, New York, 1956.

[3] Kruskal, M., "Asymptotology," *Plasma Physics,* International Atomic Energy Agency, Vienna, 1965, pp. 373-387.

[4] Ramnath, R.V., "A Multiple Time Scales Approach to the Analysis of Linear Systems," Ph.D. Dissertation, Princeton University, Princeton, N.J., 1967, also in AFFDL-TR-68-60, Wright Patterson Air Force Base, Ohio, 1968.

[5] Ramnath, R.V. and Sandri, G., "A Generalized Multiple Scales Approach to a Class of Linear Differential Equations," *Journal of Mathematical Analysis and Applications,* Vol. 28, No. 2, 1969, pp. 339-364.

[6] Ramnath, R.V., "Gravitational Perturbations of Equational Orbits," *Celestial Mechanics,* Vol. 8, 1973, pp. 85-98.

[7] Glasson, D. and Ramnath, R.V., "Asymptotic Decomposition of a Class of Large Scale Systems," *Proceedings of the IEEE-SIAM Conference on Optimization Days,* Concordia University, Montreal, 1977.

[8] Klimas, A., Ramnath, R.V., and Sandri, G., "On the Compatibility Problem in the Uniformization of Asymptotic Expansions," *Journal of Mathematical Analysis and Applications,* Vol. 32, 1970, pp. 482-504.